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Diffusion on percolating clusters

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The moments τ_k of typical diffusion times for "blind" and "myopic" ants on an arbitrary cluster are expressed exactly in terms of resistive correlations for the associated resistor network. For a diluted lattice at bond concentration p, we introduce "diffusive" susceptibilities $\chi_k(p)$ as the average over clusters of τ_k. For $p \to p_c$, where p_c is the percolation threshold, $\chi_k(p)$ diverges as $|p_c - p|^{-\gamma_k}$. We show that $\gamma_k = k\Delta_\tau - \beta$ with $\Delta_\tau = \beta + \gamma + \zeta$, where β and γ are percolation exponents and ζ is the resistance scaling exponent. Our analysis provides the first analytic demonstration that the leading exponents γ_k are the same for a wide class of models, including the two types of ants as special cases, although corrections to scaling are larger for the myopic ant than for the blind one. This class of models includes that for dilute spin waves in Heisenberg ferromagnets. Exact enumerations allow us to study universal amplitude ratios (at $p = p_c$) $\chi_{k+1}\chi_{k-1}/\chi_k^2$ as a function of continuous spatial dimension d. For $d > 6$ these ratios assume a constant value which for $k = 2$ agrees with the exact result for the Cayley tree. The χ_k have the scaling properties predicted by Gefen, Aharony, and Alexander [Phys. Rev. Lett. **50**, 77 (1983)] for anomalous diffusion.

I. INTRODUCTION

The problem of diffusion on percolating clusters has attracted much attention in the last ten years, following deGennes's proposal[1] of the "ant in the labyrinth." Using the Einstein relation between diffusion and conductivity, it has recently become clear[2] that diffusion becomes anomalous for times shorter than a typical crossover time, τ, of order $\xi^{2+\theta}$. Here $\xi \sim |p - p_c|^{-\nu}$ is the percolation correlation length for pair connectedness[3] and p is the concentration in the bond diluted lattice. Also θ is the exponent[2] describing the scaling of the diffusion coefficient $\mathcal{D}$ on the infinite cluster for p greater than the threshold value, p_c, for percolation: $\mathcal{D} \sim \xi^{-\theta}$. This exponent is related to other exponents via

$$\theta = (\mu - \beta)/\nu , \tag{1.1}$$

where $\Sigma \sim |p - p_c|^{\mu}$ and $P_\infty \sim |p - p_c|^{\beta}$ are, respectively, the conductivity and the probability per site to belong to the infinite cluster.

On length scales short compared to ξ and to the size of the cluster, and times short compared to τ, one expects anomalous diffusion, with the mean-square distance after t time steps behaving as[2] $\langle r^2 \rangle \sim t^{2/(2+\theta)}$, for $t \gg 1$, where $\langle\ \rangle$ denotes an average, for a given cluster, over all random walks of t steps (including an average over initial points). On this length scale typical clusters are self-similar, and the number of sites or bonds within a linear scale r is proportional to r^D, with the fractal dimensionality

$$D = d - \beta/\nu , \tag{1.2}$$

where d is the usual Euclidean dimensionality. Above the threshold, the infinite cluster becomes homogeneous on scales $r > \xi$, and the anomalous diffusion crosses over to a regular one, $\langle r^2 \rangle \sim \mathcal{D}t$. Thus τ is interpreted as the time to diffuse a distance $r \sim \xi$. Below the threshold there are practically no clusters of size larger than ξ. Within a cluster Γ of $s(\Gamma)$ sites and of linear size $R_\Gamma \sim s(\Gamma)^{1/D}$, we expect anomalous diffusion until $r \sim R_\Gamma$, or equivalently $t \sim R_\Gamma^{2+\theta}$, and then $\langle r^2 \rangle$ approaches a time-independent limit of order R_Γ^2.

One can also consider averages over clusters with weights appropriate to percolation. These averages, denoted $[\ ]_{\rm av}$, are defined precisely in Eq. (2.9) below. Then one has the result[2] $[\langle r^2 \rangle]_{\rm av} \sim t^{(2-\beta/\nu)/(2+\theta)}$ for $1 \ll t < \tau$. For times longer than $\tau \sim \xi^{2+\theta}$ we expect[3] $[\langle r^2 \rangle]_{\rm av}$ to approach the time-independent limit $[R_\Gamma^2]_{\rm av} \sim \xi^{2-\beta/\nu}$. One aim of the present paper is to discuss in detail quantitative determinations of the crossover time τ.

The fact that random walks can yield information on the exponent θ, and thereby on the conductivity exponent μ, led to many numerical simulations of random walks on percolating clusters.[4] Following deGennes's suggestion,[1] two main types of random walkers have been used, the "blind" ant and the "myopic" ant. The blind ant picks a random direction on the lattice and hops only if the bond in this direction is occupied. Otherwise it waits for the next time step to try again to hop. The myopic ant examines only the occupied bonds and selects randomly from one of these a direction in which to hop. Although the simulations[4] showed a similar asymptotic behavior of the two ants, there exists some controversy[5,6] concerning the way the two models approach asymptotia. Furthermore, as yet there exists no analytic formulation to show that the two ants are asymptotically the same.

In order to discuss the crossover time quantitatively,

we consider the probability $P_{ij}(t)$ for the ant to be at site i after t time steps if it started at site j. From the above discussion, the averaged quantity $[P_{ij}(t)]_{\rm av}$ is expected to exhibit a scaling behavior in terms of t/τ and r_{ij}/ξ. To simplify our study of the role of these two variables, we concentrate here on the special case of $P_{ii}(t)$, the probability to return to the origin. At $t=0$, $P_{ii}(t)\equiv 1$. For $p>p_c$ the situation is as follows. On the infinite cluster, for times $1\ll t\ll\tau$, the average $P(t)\equiv\langle P_{ii}(t)\rangle$ behaves as $P(t)\sim\langle r^2\rangle^{-D/2}\sim t^{-\bar{\bar{d}}/2}$, where $\bar{\bar{d}}$ is the fracton dimensionality:[7,8]

$$\bar{\bar{d}}=[2(\beta+\gamma)]/(2\nu+\mu-\beta)\ . \tag{1.3}$$

The same behavior is expected on finite clusters, at times $1\ll t\ll R_\Gamma^{2+\theta}$. However, for longer times the distribution on the finite cluster approaches equilibrium, and $P_{ii}(t)\to P_{ii}(\infty)\sim 1/s(\Gamma)$. To study this approach to equilibrium, we define for an arbitrary cluster Γ

$$F(\Gamma,t)=\sum_{i\in\Gamma}[P_{ii}(t)-P_{ii}(\infty)]=s(\Gamma)[P(t)-P(\infty)]\ . \tag{1.4}$$

The quantity F is expected to be of order $s(\Gamma)$ at $t=0$, and to decay to zero as $t\to\infty$, with a decay constant of order $R_\Gamma^{2+\theta}$. As discussed in detail in Sec. V, we expect the scaling behavior for $1\ll t<\tau$

$$F_{\rm av}(t)\equiv[F(\Gamma,t)]_{\rm av}=|p-p_c|^{\beta}f(t/\tau)\ , \tag{1.5}$$

with $\tau\sim\xi^{2+\theta}$, or

$$\tau=\tau_0(p_c-p)^{-\Delta_\tau}\ , \tag{1.6}$$

with $\Delta_\tau=\nu(2+\theta)$ and τ_0, the fundamental hopping time, is taken to be unity. In what follows we shall particularly be concerned with moments of $F_{\rm av}(t)$,

$$\chi_k(p)=\frac{1}{(k-1)!}\int_0^\infty dt\, t^{k-1}F_{\rm av}(t),\quad k\geq 1 \tag{1.7}$$

and we refer to these as "diffusional susceptibilities" quite analogous to "resistive susceptibilities."[9,10] From Eq. (1.5) we see that these diverge for $p\to p_c$ as

$$\chi_k(p)=A_k(p_c-p)^{(\beta-k\Delta_\tau)},\quad k\geq 1 \tag{1.8}$$

with the constant gap exponent Δ_τ. As usual in such situations,[11,12] we expect that amplitude ratios like

$$R_{kl/mn}\equiv\frac{A_kA_l}{A_mA_n},\quad k+l=m+n \tag{1.9}$$

will be universal.

The exponent Δ_τ is related to two other gap exponents, $\Delta=D\nu=\beta+\gamma=d\nu-\beta$, the gap exponent for the cluster size distribution,[3,12] and ζ, the gap exponent for moments of the resistance.[9,10] These are defined by

$$[s(\Gamma)^k]_{\rm av}\sim(p_c-p)^{(\beta-k\Delta)},\quad k\geq 1 \tag{1.10}$$

and

$$\left[\sum_{i,j\in\Gamma}R_{ij}^k/s(\Gamma)\right]_{\rm av}\sim(p_c-p)^{(-\gamma-k\zeta)}\ , \tag{1.11}$$

respectively. To obtain a relation for Δ_τ note that for $p>p_c$ the conductance, g, of a block of size ξ is given by

$$g=\Sigma\xi^{d-2}\ , \tag{1.12}$$

and by Eq. (1.11) we may write $g\sim\xi^{-\zeta/\nu}$. Also $\Sigma\sim\xi^{-\mu/\nu}$ which by Eq. (1.1) is $\Sigma\sim\xi^{-\theta-\beta/\nu}$. Putting these scaling results into Eq. (1.12) yields $2+\theta=D+\zeta/\nu$. Consequently

$$\Delta_\tau=\Delta+\zeta\ . \tag{1.13}$$

The scaling result (1.13) thus seems to relate these two previously studied problems. Indeed, Essam and Bhatti[13] recently established the relation between the diffusional relaxation time susceptibility $\chi_1(p)$ of the blind ant and the resistive susceptibility of the associated resistor network [both of which scale as $(p_c-p)^{-\gamma-\zeta}$]. However, no analogous relation has been presented either for the myopic ant or for the higher moments $\chi_k(p)$. The purposes of the present paper are (a) to obtain general relations between the two ant models and the associated resistor networks, (b) to use these relations to obtain exact results on the Cayley tree and in one dimension for the amplitude ratios of Eq. (1.6), (c) to study rigorously the presumed asymptotic equivalence of the blind and myopic ants, to discuss possible differences in the corrections to their dominant behavior, and to find a more general problem of which both ants are special cases, and (d) to obtain low-concentration series for $\chi_k(p)$ on hypercubic lattices in arbitrary spatial dimension, d, and to use these to find the gap exponents Δ_τ and the universal amplitude ratios $R_{kl/mn}$.

Briefly, this paper is organized as follows. In Sec. II we define the diffusional susceptibilities in terms of averages over clusters of the moments of the relaxation rates. Here we derive exact relations between these quantities and certain resistance correlations for the blind ant. For the myopic ant the results given here are derived in Appendix B. In Sec. III we give exact results for diffusional susceptibilities in one dimension which allow us to compare both ants with respect not only to their asymptotic behavior but also to the amplitudes of the leading corrections thereto. Here we also quote exact results for the Cayley tree which are obtained in Appendix A. In Sec. IV we analyze series results for the diffusional susceptibilities to obtain associated exponents and amplitude ratios as a function of spatial dimensionality d for $d\geq 2$. Our results are generalized in Sec. V to a wider class of diffusion models and are discussed within a general scaling framework. Our conclusions are summarized briefly in Sec. VI.

II. FORMULATION AND RELATIONS BETWEEN DIFFUSIONAL AND RESISTIVE SUSCEPTIBILITIES

A. General

The diffusion on an arbitrary cluster, Γ, can be described by the continuous-time master equation,

$$\frac{d}{dt}P_{ij}(t)=\sum_k [W_{ki}P_{kj}(t)-W_{ik}P_{ij}(t)]$$

$$\equiv -\sum_k M_{ik}P_{kj}(t) , \tag{2.1}$$

where W_{ki} is the probability per unit time that the particle hops from site k to site i, with the initial condition that $P_{ij}(0)=\delta_{i,j}$. For the blind ant, we take[14] $W_{ki}=\gamma_{ki}/2$, where γ_{ki} is unity if i and k are nearest-neighboring sites connected by an occupied bond, and is zero otherwise. Thus for the blind ant we set $M=M^B$ with

$$M^B_{ik}=(z_i\delta_{i,k}-\gamma_{ik})/2 , \tag{2.2}$$

where δ is the Kronecker delta and $z_i=\sum_k\gamma_{ik}$ is the number of occupied bonds intersecting site i. For the myopic ant $W_{ki}=\gamma_{ik}/z_k$, and we set $M=M^M$ with

$$M^M_{ik}=\delta_{i,k}-\gamma_{ik}/z_k . \tag{2.3}$$

For the blind ant, the matrix M^B is symmetric and we can solve Eq. (2.1) as

$$P_{ij}(t)=\sum_n \phi_i^{(n)}\phi_j^{(n)}e^{-\lambda_n t} , \tag{2.4}$$

in terms of the eigenvectors $\phi^{(n)}$ (assumed normalized) and the eigenvalues λ_n of the dynamical matrix

$$\sum_j M^B_{ij}\phi_j^{(n)}=\lambda_n\phi_i^{(n)} . \tag{2.5}$$

The equilibrium distribution, $P_{ij}(\infty)$, is given in terms of the eigenvector with zero eigenvalue ($\lambda_0=0$) as

$$P_{ij}(\infty)=\phi_i^{(0)}\phi_j^{(0)}\equiv 1/s(\Gamma) . \tag{2.6}$$

The existence of a zero eigenvalue follows from the fact that $\sum_k M_{ik}=0$ for our models. Using the orthogonality of the eigenvectors we thus find that

$$F(\Gamma,t)\equiv\sum_{i\in\Gamma}[P_{ii}(t)-P_{ii}(\infty)]=\sum_n{}' e^{-\lambda_n(\Gamma)t} , \tag{2.7}$$

where the prime on the summation indicates omission of the term with $\lambda_0=0$. Thus

$$\chi^B_k(p)=\left[\sum_n{}'\lambda_n(\Gamma)^{-k}\right]_{\rm av}\equiv[s(\Gamma)\tau_k(\Gamma)]_{\rm av} , \tag{2.8}$$

where the superscript on χ_k indicates the type of ant involved, and τ_k may be identified as the kth power of a typical relaxation time for diffusion on the cluster Γ. Here for any cluster property $X(\Gamma)$, its cluster average $[X(\Gamma)]_{\rm av}$ is defined by

$$[X(\Gamma)]_{\rm av}=\sum_\Gamma W(\Gamma)s(\Gamma)X(\Gamma)p^{b(\Gamma)}(1-p)^{t(\Gamma)} , \tag{2.9}$$

where $W(\Gamma)$ is the weak embedding constant (the number of times per site that a cluster topologically equivalent to Γ can occur), $b(\Gamma)$ is the number of bonds in the cluster Γ, $t(\Gamma)$ is the number of perimeter bonds of Γ, and the sum is carried over all topologically inequivalent clusters Γ. (This definition is such that $[s(\Gamma)]_{\rm av}$ is the percolation susceptibility.) Equation (2.8) indicates that $\chi_k(p)$ is interpretable as the average of the kth moment of the relaxation times λ_n^{-1}.

For the myopic ant, the matrix M^M is not symmetric. However, one can still identify its eigenvalues μ_n, and use their moments to define a diffusional susceptibility via

$$\chi^M_k(p)=\left[\sum_n{}'\mu_n(\Gamma)^{-k}\right]_{\rm av} . \tag{2.10}$$

B. Green's functions

We introduce the Laplace transform of P_{ij}:

$$G_{ij}(\omega)=\int_0^\infty dt\, e^{-\omega t}P_{ij}(t)=(M+\omega I)^{-1}_{ij} , \tag{2.11}$$

so that the Green's function $G_{ij}(\omega)$ is the matrix inverse of $M_{ij}+\omega\delta_{i,j}$. For the blind ant,

$$G_{ij}(\omega)=\sum_n \phi_i^{(n)}\phi_j^{(n)}(\omega+\lambda_n)^{-1} . \tag{2.12}$$

In particular, the Laplace transform of $F(\Gamma,t)$ is

$$\tilde F(\Gamma,\omega)=\sum_n{}'(\omega+\lambda_n)^{-1}={\rm Tr}\hat G(\omega) , \tag{2.13}$$

where $\hat G$ is the part of G which is nonsingular as $\omega\to 0$:

$$\hat G_{ij}(\omega)=G_{ij}(\omega)-\phi_i^{(0)}\phi_j^{(0)}/\omega , \tag{2.14}$$

and the Laplace transform of $F_{\rm av}(t)$ is

$$\tilde F_{\rm av}(\omega)\equiv[{\rm Tr}\hat G(\omega)]_{\rm av} , \tag{2.15}$$

which asymptotically (for ω small and τ large) depends on ω only in the combination $(\omega\tau)$. All the diffusional susceptibilities $\chi_k(p)$ can now be obtained from derivatives of $\tilde F_{\rm av}(\omega)$ at $\omega=0$:

$$\chi_k(p)=[{\rm Tr}\hat G(0)^k]_{\rm av} . \tag{2.16}$$

C. Blind ant and resistive susceptibilities

Kirchhoff's equations for a resistor network can be written as

$$2\sum_r M^B_{kr}V_r=I^{\rm ext}_k , \tag{2.17}$$

where $I^{\rm ext}_k$ is the externally imposed current at site k and M^B is given in Eq. (2.2). To find the resistance R_{ij} between sites i and j we set $I^{\rm ext}_k=I_0(\delta_{i,k}-\delta_{j,k})$, and get $R_{ij}=(V_i-V_j)/I_0$. Using $V_k=\sum_r[(2M^B)^{-1}]_{kr}I^{\rm ext}_r$, we obtain

$$2R_{ij}=\lim_{\omega\to 0}G_{ii}+G_{jj}-2G_{ij} ,$$

$$=\hat G_{ii}+\hat G_{jj}-2\hat G_{ij} , \tag{2.18}$$

where from now on, $\hat G(\omega)$ is to be evaluated at $\omega=0$. Since $\phi^{(n)}$ is orthogonal to $\phi^{(0)}$ for $n\neq 0$, we see that $\sum_i\hat G_{ij}=\sum_j\hat G_{ij}=0$. Thus from Eq. (2.18) we get

$$2\sum_j R_{ij}\equiv 2s(\Gamma)\bar R_i=s(\Gamma)\hat G_{ii}+\sum_j\hat G_{jj} , \tag{2.19a}$$

$$2\sum_{i,j}R_{ij}\equiv 2s^2(\Gamma)\bar R=2s(\Gamma)\sum_j\hat G_{jj} . \tag{2.19b}$$

Solving Eqs. (2.18), (2.19a), and (2.19b) for $\hat{G}_{ij}$ we find

$$\hat{G}_{ij} = -R_{ij} + \bar{R}_i + \bar{R}_j - \bar{R} \ . \tag{2.20}$$

Thus Eq. (2.16) yields

$$\chi_1(p) = [s(\Gamma)\bar{R}]_{\rm av} = \left[\sum_{i,j \in \Gamma} R_{ij}/s(\Gamma) \right]_{\rm av} , \tag{2.21}$$

which is exactly the resistive susceptibility of Eq. (1.11), as also found by Essam and Bhatti.[13]

Similarly, for an arbitrary cluster Γ we use the expression of Eq. (2.20) for $\hat{G}_{ij}$ in terms of resistances to evaluate $\chi_k(p)$ for $k > 1$ by Eq. (2.16) and obtain

$$\chi_2(p) = \left[\sum_{i,j} R_{ij}^2 - 2s(\Gamma) \sum_i \bar{R}_i^2 + s^2(\Gamma)\bar{R}^2 \right]_{\rm av} , \tag{2.22}$$

$$\chi_3(p) = \left[s^3(\Gamma)\bar{R}^3 - 3s^2(\Gamma) \sum_i \bar{R}_i^2 \bar{R} + 3s(\Gamma) \sum_{i,j} \bar{R}_i R_{ij} \bar{R}_j - \sum_{i,j,k} R_{ij} R_{jk} R_{ki} \right]_{\rm av} , \tag{2.23}$$

and generally

$$\chi_k(p) = \left[s^{-k}(\Gamma) \sum_{(i,j)} \prod_{m=1}^{k} (R_{i_m, j_m} - R_{i_m, i_{m+1}}) \right]_{\rm av} , \tag{2.24}$$

where the sum is over sites $i_1, i_2, \ldots, i_k$ and $j_1, j_2, \ldots, j_k$ all limited to the cluster Γ, and we define $i_{k+1} = i_1$.

From Eq. (2.24) we see that all the $\chi_k(p)$'s can be calculated using tabulations of R_{ij} for each cluster, without explicit solutions of the eigenvalue problem for the λ's. These relations are also essential in obtaining exact solutions for the $\chi_k(p)$'s on the Cayley tree, as done in Appendix A.

D. Myopic ant

For the myopic ant, the Green's function is

$$G_{ij}^M(\omega) = (M^M + \omega I)_{ij}^{-1} , \tag{2.25}$$

with M^M given by Eq. (2.3). Thus for an arbitrary cluster Γ we set

$$\tilde{F}^M(\Gamma, \omega) = {\rm Tr} G^M(\omega) - \omega^{-1} \equiv \sum_n{}' (\omega + \mu_n)^{-1} , \tag{2.26a}$$

so that

$$\tilde{F}_{\rm av}^M(\omega) = \left[\sum_n{}' (\omega + \mu_n)^{-1} \right]_{\rm av} = \sum_k (-1)^k \chi_{k+1}(p) \omega^k . \tag{2.26b}$$

It is convenient to note that

$$M^B = M^M Z , \tag{2.27}$$

where Z is the diagonal matrix

$$Z_{ij} = \delta_{i,j} z_i / 2 . \tag{2.28}$$

Thus,

$$G^M(\omega) = Z(M^B + Z\omega)^{-1} . \tag{2.29}$$

To subtract $1/\omega$, we note that the right and left eigenvectors of M^M with eigenvalue $\mu_0 = 0$ are $Z \mid 0\rangle$ and $\langle 0 \mid$, respectively, where $\mid 0\rangle$ is a vector with components $\phi_i^{(0)} = s^{-1/2}(\Gamma)$, so that $M^B \mid 0\rangle = 0$. Thus

$$\omega^{-1} = \langle 0 \mid G^M(\omega) Z \mid 0 \rangle / \langle 0 \mid Z \mid 0 \rangle . \tag{2.30}$$

For $\omega = 0$ we therefore write Eq. (2.26a) as

$$\tilde{F}^M(\Gamma, 0) = \lim_{\omega \to 0} {\rm Tr} Z G^B(\omega) - \langle 0 \mid Z G^B(\omega) Z \mid 0 \rangle / \langle 0 \mid Z \mid 0 \rangle \tag{2.31a}$$

$$= {\rm Tr} \hat{G} Z - \langle 0 \mid Z \hat{G} Z \mid 0 \rangle / \langle 0 \mid Z \mid 0 \rangle \tag{2.31b}$$

$$= \sum_{i \in \Gamma} \hat{G}_{ii} z_i / 2 - \sum_{i,j \in \Gamma} \hat{G}_{ij} z_i z_j \Big/ \left[2 \sum_{i \in \Gamma} z_i \right] , \tag{2.31c}$$

where $\hat{G}$ is as in Eq. (2.14). Using Eq. (2.20) we have that

$$\chi_1^M(p) = \tfrac{1}{2} \left[\sum_{i,j \in \Gamma} R_{ij} z_i z_j \Big/ \sum_{i \in \Gamma} z_i \right]_{\rm av} = \tfrac{1}{4} \left[\sum_{i,j \in \Gamma} R_{ij} z_i z_j / b(\Gamma) \right]_{\rm av} , \tag{2.32}$$

where we have used $\sum_i z_i / 2 = b(\Gamma)$, the number of bonds on the cluster. Replacing each z_i by

$$z_i = \langle z_i \rangle + \delta z_i = [2b(\Gamma)/s(\Gamma)] + \delta z_i , \tag{2.33}$$

we see that

$$\chi_1^M(p) = \left[\sum_{i,j} R_{ij} \frac{b(\Gamma)}{s^2(\Gamma)} + \sum_{i,j} R_{ij} \frac{(\delta z_i)}{s(\Gamma)} + \tfrac{1}{4} \sum_{i,j} R_{ij} \frac{(\delta z_i \delta z_j)}{b(\Gamma)} \right]_{\rm av} . \tag{2.34a}$$

For large clusters the ratio $b(\Gamma)/s(\Gamma)$ approaches a constant, which we denote $b(\infty)/s(\infty)$. This ratio depends on the density of small loops and therefore will be sensitive to the range of connectivity of percolation. Accordingly, this constant is not universal. Nevertheless, we expect the first term in Eq. (2.34a) to have the same leading critical behavior as $\chi_1(p)$ in Eq. (2.21). The other two terms in Eq. (2.34a) involve local fluctuations in the connectedness parameter, $\delta z_i = \sum_l (\gamma_{il} - \langle \gamma_{il} \rangle)$. As such, we expect them to yield higher-order correlation functions, and therefore less divergent contributions, so that

$$\chi_1^M(p)\sim\frac{b(\infty)}{s(\infty)}\chi_1^B(p)\ .$$

Similar statements can be made for the higher-order diffusional susceptibilities of the two ants. For this purpose we have derived, as discussed in Appendix B, the expressions for the myopic ant:

$$4\chi_2^M(p)=\left[\sum_{i,j}R_{ij}^2z_iz_j+\left[\sum_{i,j}z_iR_{ij}z_j\Big/\sum_i z_i\right]^2-2\sum_{i,j,k}R_{ij}R_{jk}z_iz_jz_k\Big/\sum_i z_i\right]_{\rm av}, \tag{2.34b}$$

and

$$8\chi_3^M(p)=\left[-\sum_{i,j,k}R_{ij}R_{jk}R_{ki}z_iz_jz_k+3\sum_{i,j,k,l}R_{ij}R_{jk}R_{kl}z_iz_jz_kz_l\Big/\sum_i z_i-3\left[\sum_{i,j}R_{ij}z_iz_j\right]\left[\sum_{i,j,k}R_{ij}R_{jk}z_iz_jz_k\right]\Big/\left[\sum_i z_i\right]^2+\left[\sum_{i,j}R_{ij}z_iz_j\Big/\sum_i z_i\right]^3\right]_{\rm av}, \tag{2.34c}$$

from which we guess that the general result may be

$$\chi_k^M(p)=\left[\sum_{(i,j)}\prod_{m=1}^{k}\frac{z_{i_m}}{2}\left[R_{i_m,i_{m+1}}/s(\Gamma)-R_{i_m,j_m}z_{j_m}\Big/\sum_r z_r\right]\right]_{\rm av} \tag{2.35}$$

in the notation of Eq. (2.24). Comparing Eqs. (2.34) or (2.35) with the analogous results for the blind ant we expect the general result

$$\chi_k^M(p)\sim\chi_k^B(p)\left[\frac{b(\infty)}{s(\infty)}\right]^k. \tag{2.36}$$

III. EXACT SOLUTIONS FOR ONE DIMENSION AND CAYLEY TREE

In one dimension the model of the blind ant becomes exactly soluble. When $|i-k|=1$, M_{ik} in Eq. (2.2) becomes $-\frac{1}{2}$ and $M_{ii}=1$ for all sites but the ends. The eigenvalues λ_n of Eq. (2.5) are $1-\cos(n\pi/N)$, where n ranges from 0 to $N-1$, with $N=s(\Gamma)$, so that

$$\chi_k(p)=\sum_{s=2}^{\infty}\left[\sum_{n=1}^{s-1}[1-\cos(n\pi/s)]^{-k}\right]sp^{s-1}(1-p)^2 \tag{3.1a}$$

$$=\sum_{s=2}^{\infty}\chi_k^{(s)}sp^{s-1}(1-p)^2\ . \tag{3.1b}$$

One can evaluate the sum in the large parentheses using partial fractions[15]

$$\chi_1^{(N)}=\sum_{n=1}^{N-1}[1-\cos(n\pi/N)]^{-1}=(N^2-1)/3\ , \tag{3.2a}$$

$$\chi_2^{(N)}=\sum_{n=1}^{N-1}[1-\cos(n\pi/N)]^{-2}=(2N^4+5N^2-7)/45\ , \tag{3.2b}$$

$$\chi_3^{(N)}=\sum_{n=1}^{N-1}[1-\cos(n\pi/N)]^{-3}=(8N^6+21N^4+42N^2-71)/945\ , \tag{3.2c}$$

which leads to the results

$$\chi_1^B(p)=2(1-p)^{-2}[1-(1-p)\cdots]\ , \tag{3.3a}$$

$$\chi_2^B(p)=\tfrac{16}{3}(1-p)^{-4}[1-2(1-p)\cdots]\ , \tag{3.3b}$$

$$\chi_3^B(p)=\tfrac{128}{3}(1-p)^{-6}[1-3(1-p)\cdots]\ , \tag{3.3c}$$

or generally

$$\chi_k^B(p)=D_k(1-p)^{-2k}[1-k(1-p)\cdots],\quad k\geq 1 \tag{3.3d}$$

$$=p/(1-p),\quad k=0 \tag{3.3e}$$

where we have kept the dominant term and only the first correction term and also D_k is a constant. Comparing Eqs. (3.3d) and (1.8) we see that $\beta=0$ and $\Delta_r=2$ in one dimension. Also, Eq. (3.3e) gives $\gamma=1$. These results therefore imply that $\zeta=\Delta_r-\beta-\gamma=1$, which is clearly the correct value in one dimension for the scaling exponent of the resistance $R(x,x')\sim|x-x'|^{\tilde{\zeta}}$, with $\tilde{\zeta}=\tilde{\zeta}\nu=\tilde{\zeta}$.

The difference between the model of the blind ant and the model of the myopic ant in one dimension lies only in the boundary and we expect this difference to disappear in the limit $N\to\infty$. In fact, one can establish the relation

$$\chi_k^{(N),M}=\chi_k^{(N-1),B}+2^{-k} \tag{3.4}$$

between the myopic and blind ants. Use of this relation leads to the analogs of Eq. (3.3), but now for the myopic ant:

$$\chi_k^M(p)=D_k(1-p)^{-2k}\left[1-\left[k+\frac{2k}{2k+1}\right](1-p)\right],\quad k\geq 1\ . \tag{3.5}$$

In order to evaluate $\chi_k(p)$ exactly on a Cayley tree, it is essential to use the relations obtained in Sec. II between the moments of the typical diffusion time and the correlations of resistances. This calculation, however, is rather cumbersome, and we give it in detail in Appendix A. We quote here the results for the dominant asymptotic behavior as $p\to p_c$:

$$\chi_k(p)=C_k\frac{\sigma+1}{\sigma}\left[\frac{\sigma-1}{\sigma}\right]^k\frac{1}{(1-\sigma p)^{3k-1}}\,,$$

$$k\geq 1\quad(\sigma>1)\qquad(3.6)$$

where $\sigma+1$ is the coordination number and for the blind and myopic ants $C_1=1$, $C_2=\frac{4}{5}$, $C_3=\frac{47}{10}$, and so forth. Thus, as expected, we have $\beta=1$ and $\Delta_\tau=3$ on a Cayley tree corresponding to dimensions above the upper critical dimension, which for this problem is 6.[16,17]

Though the above formulas give the same leading behavior for the universal amplitude ratios for the two ants, we find that the amplitudes of the corrections to scaling, which are analytic on the Cayley tree, are not the same for the two ants. For example, from Eqs. (3.4) and (3.5) we see that in one dimension the relative amplitude of the correction for the myopic ant is larger than that for the blind ant.

IV. SERIES RESULTS

For $d>1$ these models cannot be solved exactly. One notes that if the sum in Eq. (2.9) is expanded in powers of p, then the nth order term involves only clusters with up to n bonds. We calculated the first 11 terms for $\chi_k(p)$ for the two models in two ways. Firstly, one can use the definition of $\chi_k(p)$ in Eq. (2.8) or (2.10) in terms of the eigenvalues found by diagonalizing the matrix M and summing over the kth power of their inverses. Alternatively, one can use the relations obtained in Sec. II between $\chi_k(p)$ and the resistance correlations obtained by calculating the resistances between all pairs of points in the cluster. These two methods agreed, as expected, but the second one involved only solving sets of linear equations rather than diagonalizing a matrix, so from a numerical point of view it was preferable. Table I contains the series coefficients for $\chi_1(p)$ for the myopic ant; that for the blind ant is equivalent to the resistive susceptibility which has already been published[9] up to tenth order, so we give in Table I also the eleventh-order coefficients for the blind ant. We analyzed the various series using two methods and the results are given in Table II (Refs. 18–20) for $1\leq d\leq 6$. The first was to apply the nonhomogeneous Differential Padé method[21] to the series $\chi_k(p)$. In this way we obtained, for each series, a large number ($\sim$40 for the blind ant series) of estimates for the critical concentration p_c and the exponent γ_k. Then by interpolating to the known values[3] of p_c, we obtained estimates for the exponents γ_k. We estimated the error as the error of the interpolation procedure. In order to obtain estimates for ζ and Δ_τ one has to subtract γ from the resulting exponents, and we used γ obtained from the same analysis applied to the percolation susceptibility series,[19] to reduce any systematic errors. The second method of analysis was to first divide the coefficients of the series for $\chi_1(p)$ by the

TABLE I. (a) The myopic ant diffusional susceptibility $\chi_1(p)=dp+(5d^2-2.5d)p^2+(18d^3-17\frac{2}{3}d^2+4\frac{1}{3}d)p^3+(51d^4-75\frac{2}{3}d^3+12\frac{11}{12}d^2+18\frac{5}{12}d)p^4+\sum_{l,m}a_{l,m}p^l d^m$. (b) The eleventh-order coefficients $a_{11,m}$ of the blind ant diffusional susceptibility. Overbars indicate the last digit is repeated.

(a)

l,m	$a_{l,m}$	l,m	$a_{l,m}$	l,m	$a_{l,m}$
5,5	133.2	5,4	−264.9$\bar{3}$	5,3	59.3$\bar{6}$
5,2	194.1$\bar{6}$	5,1	−113.1$\bar{3}$	6,6	328.9$\bar{3}$
6,5	−819.3$\bar{5}$	6,4	283.6$\bar{1}$	6,3	480.30$\bar{5}$
6,2	188.53$\bar{8}$	6,1	−451.3$\bar{6}$	7,7	783.7$\bar{3}$
7,6	−2345.670	7,5	1189.565	7,4	1128.240
7,3	3656.597	7,2	−8638.675	7,1	4238.876
8,8	1820.0$\bar{6}$	8,7	−6359.816	8,6	4394.351
8,5	2492.981	8,4	5886.983	8,3	4225.524
8,2	−45 914.831	8,1	33 469.408$\bar{3}$	9,9	4146.652
9,8	−16 567.093	9,7	14 745.374	9,6	47 218.660
9,5	6340.323	9,4	126 740.765	9,3	−558 601.811
9,2	700 313.615	9,1	−281 823.024	10,10	9 308.270
10,9	−41 850.595	10,8	46 020.551	10,7	5 895.935
10,6	−22 96.916	10,5	259 345.072	10,4	−210 346.077
10,3	−2 919 332.319	10,2	6193 406.513	10,1	−3 340 132.767
11,11	20 649.406	11,10	−103 178.192	11,9	135 866.485
11,8	−54 527.893	11,7	−43 070.489	11,6	520 175.619
11,5	3 690 516.518	11,4	−31 428 592.806	11,3	72 988 106.626
11,2	−69 851 501.174	11,1	24 076 501.464		

(b)

m	$a_{11,m}$	m	$a_{11,m}$	m	$a_{11,m}$
11	11 264	10	−56 320	9	72 960
8	832	7	−25 525.$\bar{3}$	6	287 539.339
5	2 130 349.359	4	−18 009 115.630	3	41 896 738.401
2	−40 219 644.741	1	13 910 933.606		

TABLE II. Estimates of ζ and Δ_τ obtained in this paper, compared to available data.

d	ζ^a	ζ^b	ζ^c	$\Delta_\tau{}^d$	$\Delta_\tau{}^e$	$(\beta+\gamma+\zeta)^f$
1	1 (exact)	1 (exact)	1 (exact)	2 (exact)	2 (exact)	2 (exact)
2	1.32±0.08	1.30±0.06	1.30±0.02	3.5 ±0.2	3.7 ±0.2	3.83±0.02
3	1.13±0.04	1.06±0.03	1.11±0.05	3.2 ±0.2	3.30±0.05	3.34±0.10
4	1.04±0.03	1.04±0.01	1.05±0.02	3.07±0.07	3.10±0.03	3.15±0.06
5	1.04±0.03	1.02±0.01	1.02±0.02	3.0 ±0.1	3.04±0.03	3.05±0.07
6	1 (exact)	1 (exact)	1 (exact)	3 (exact)	3 (exact)	3 (exact)

[a]Estimates obtained by applying the nonhomogeneous differential Padé method (DPM) to the series for $\chi_1(p)$ for the blind ant and subtracting γ.

[b]Estimates obtained by applying DPM to the series constructed by dividing the coefficients of $\chi_1(p)$ by the percolation series term by term.

[c]For a detailed list of the values available in the literature see Meir *et al.* (Ref. 18).

[d]Estimates obtained by applying DPM to the series of the kth moment diffusional susceptibility for the blind ant and subtracting the exponents of the $(k-1)$st moment for $k=2$, 3, and 4 and averaging.

[e]Estimates obtained by applying DPM to the series constructed by dividing the kth moment diffusional susceptibility for the blind ant by the $(k-1)$st moment term by term for $k=2$, 3, 4, and 5 and averaging.

[f]Values of $\beta+\gamma$ are taken from Adler *et al.* (Ref. 19) for $d>2$, while for $d=2$ we took $\beta+\gamma=\frac{91}{36}$ as given by de Nijs (Ref. 20).

percolation series[19] term by term and then analyze the resulting series using the above methods. The resulting series diverges at $p=1$ with an exponent that is the difference of the exponents of the two series in question plus one.[22] Using this procedure we get an estimate for the exponent $(\zeta+1)$ which is unbiased by the values of p_c and γ. In order to get an estimate for the gap exponent, Δ_τ, we divided each term in the series for the kth moment, $\chi_k(p)$, by the corresponding term in the $(k-1)$st moment and the new series gives directly an unbiased estimate for $(\Delta_\tau+1)$.

The series for the blind ant are very well behaved. The series for the myopic ant are less well behaved, but we find within our uncertainty the same values as for the blind ant. One sees from Table II that, as expected from the scaling arguments of Sec. II, the gap exponent, Δ_τ, is equal within our uncertainties to $\beta+\gamma+\zeta$.

Furthermore, one can check the universality also by comparing amplitude ratios. We used a newly developed method[23] to get estimates for the universal quantities

$$S_{ij/kl}=\frac{A_i A_j}{A_k A_l}\frac{\Gamma(\gamma_k)\Gamma(\gamma_l)}{\Gamma(\gamma_i)\Gamma(\gamma_j)}=R_{ij/kl}\frac{\Gamma(\gamma_k)\Gamma(\gamma_l)}{\Gamma(\gamma_i)\Gamma(\gamma_j)}\ , \quad (4.1)$$

where γ_i is the exponent that describes the divergence of the ith moment of the typical time $[\chi_i(p)]$, A_i is the corresponding amplitude, and Γ is the usual gamma function. Results for $S_{13/22}$ and $S_{14/23}$ are presented in Table III, together with the exact results for high dimension for $S_{13/22}$ and show again the same values for both ants.

To verify Eq. (2.36) we determined the ratio $[b]_{\rm av}/[s]_{\rm av}$ by dividing term by term the low-concentration series[19] for these quantities. We compared this ratio to the ratio obtained by dividing $\chi_k^M(p)$ by $\chi_k^B(p)$ for $k=1$, 2, and 3, similarly determined. These two ratios agreed to within 10%.

V. GENERALIZATIONS AND DISCUSSION

A. Generalizations

We may generalize the model of Sec. II by allowing the hopping rate for the occupied bonds (i,j) to be a local variable σ_{ij}. Thus we introduce the "generalized blind" (GB) ant for which we set $M=M^{\rm GB}$ with

$$M_{ik}^{\rm GB}=\tfrac{1}{2}\left[\sum_j \sigma_{ij}\delta_{i,k}-\sigma_{ik}\right]\ , \quad (5.1)$$

where $\sigma_{ij}=\sigma_{ji}$ is nonzero only for occupied bonds. The

TABLE III. Estimates of amplitude ratios of Eq. (4.1) for the two ants in general dimensions.

$d=$	1	2	3	4	5	>6
$S_{13/22}$						
Blind:	$\frac{9}{10}$ (exact)	0.65±0.15	0.75±0.06	0.81±0.02	0.84±0.02	$\frac{47}{56}$ (exact)
Myopic:	$\frac{9}{10}$ (exact)	0.62±0.12	0.73±0.03	0.79±0.02	0.82±0.02	$\frac{47}{56}$ (exact)
$S_{24/33}$						
Blind:	$\frac{6}{7}$ (exact)	0.63±0.06	0.67±0.05	0.69±0.04	0.71±0.03	0.72±0.03[a]
Myopic:	$\frac{6}{7}$ (exact)	0.63±0.03	0.67±0.02	0.69±0.02	0.69±0.02	0.71±0.02[a]

[a]Estimates for $d=20$.

blind ant considered in Sec. II corresponds to taking $\sigma_{ij}=\gamma_{ij}$ for nearest-neighbor bonds.[14] The relations expressing $\chi_k(p)$ in terms of resistance correlations remain valid, but now the associated network consists of conductances σ_{ij}.

We now consider the relationship between the generalized diffusion described by Eq. (5.1) and the equations for the spin-wave energies, ω_α, of a bond diluted Heisenberg ferromagnet where $J_{ij}=J_{ji}$ is nonzero only for occupied bonds. The spin-wave energies are the eigenvalues of the equations

$$\omega S_i^+ = 2\sum_j J_{ij}(S_j^z S_i^+ - S_i^z S_j^+) , \tag{5.2}$$

where S_i^z is the value of the z component of the spin of site i in the ground state with the net magnetization aligned along the $+z$ axis. Supposing the system to be a ferromagnet, $S_i^z=S_i$, where S_i is the magnitude of the spin on site i. One sees that the spin-wave equations have a nonsymmetric dynamical matrix and are not of the form of the GB ant of Eq. (5.1). Thus we next consider the generalized myopic (GM) ant, for which we set $M=M^{\rm GM}$ with

$$M^{\rm GM}=M^{\rm GB}Z^{-1} , \tag{5.3}$$

where Z is a diagonal matrix which we write as

$$Z_{ij}=\delta_{i,j}\xi_i , \tag{5.4}$$

where the ξ_i are arbitrary. In the derivation of Eq. (2.32) one sees that we nowhere used the fact that z_i was a local coordination number. Thus for the GM ant we may continue to use Eq (2.32) with z_i replaced by ξ_i and, as for the GB ant, with R_{ij} calculated for the network with given conductances σ_{ij}. To obtain the spin-wave equations (5.2) we set $\xi_i=S_i^z$ and $\sigma_{ij}=4J_{ij}S_i^zS_j^z$. Thus the spin-wave equations of motion are isomorphic to the master equation for diffusion of the GM ant. The relation (2.32) is seen to hold not only for a ferromagnet, but also for an arbitrary *ferri*magnetic cluster Γ, providing only that it has a nonzero net moment: $\sum_i\xi_i=\sum_{i\in\Gamma}S_i^z\neq 0$. In general, however, unless the spins values of different sublattices are irrational, a randomly diluted ferrimagnet will have some clusters with zero total spin. Since these cannot be treated by the present formulation, we do not treat diluted ferrimagnets here.

B. Scaling arguments

Our results can be phrased in terms of the spin-wave dynamic susceptibility, $\chi(q,\omega)$, which is written in the hydrodynamic limit, $q\xi\ll 1$, as

$$\chi(q,\omega)=\frac{\langle M\rangle}{\omega-D_q q^2} , \tag{5.5}$$

where $\langle M\rangle$ is the magnetization (at zero temperature) and D_q the dispersion constant. Taking account of the scaling behavior of ω and q, we deduce that

$$\chi(q,\omega)=\frac{\langle M\rangle}{\omega}f(\omega\tau;q\xi) , \tag{5.6}$$

where f is an unspecified scaling function. A consequence of this form is that $D_q\tau/\xi^2$ scales like a constant or

$$D_q\sim(p-p_c)^{-2\nu+\Delta_\tau}=(p-p_c)^{\nu\theta}\sim\xi^{-\theta} . \tag{5.7}$$

This is equivalent to an old result:[24]

$$D_q(p)/D_q(1)=\Sigma(p)P_\infty(1)/\Sigma(1)P_\infty(p) .$$

We now consider the averaged density of states per cluster, $[\sum_i\rho_{ii}(\omega)]_{\rm av}$, where

$$\rho_{ii}(\omega)=\lim_{\epsilon\to 0}[\pi^{-1}\,{\rm Im}G_{ii}(\omega-i\epsilon)] .$$

[This quantity differs from the average density of states *per site* by inclusion in the average of a factor $s(\Gamma)$.] In the fracton picture one has

$$\left[\sum_i\rho_{ii}(\omega)\right]_{\rm av}\sim|p-p_c|^{-\gamma}\omega^{\bar{\bar{d}}/2-1} ,$$
$$p\to p_c,\ \tau^{-1}\ll\omega<1 \tag{5.8a}$$

or more generally

$$\left[\sum_i\rho_{ii}(\omega)\right]_{\rm av}\sim|p-p_c|^{-\gamma-\phi}f(\omega\tau) , \tag{5.8b}$$

where $f(x)$ is an unspecified scaling function. The dependence on frequency in Eq. (5.8a) is that usually associated with fractons.[7] The prefactor $|p-p_c|^{-\gamma}$ is included because the total number of modes in a cluster of $s(\Gamma)$ sites is equal to $s(\Gamma)$. The low-frequency cutoff is introduced because for $p<p_c$ there are essentially no clusters large enough to have frequencies below the cutoff. However, the frequency spectrum includes localized modes with frequencies of order unity. Using this assumed form we may estimate the averaged inverse frequency moments as

$$\chi_k(p)\sim\left[\int_{\tau^{-1}}^1 d\omega\sum_{i\in\Gamma}\rho_{ii}(\omega;\Gamma)\omega^{-k}\right]_{\rm av} \tag{5.9a}$$

$$\sim|p-p_c|^{-\gamma}\int_{\tau^{-1}}^1\omega^{(\bar{\bar{d}}/2-k-1)}d\omega . \tag{5.9b}$$

For $k<\bar{\bar{d}}/2$, i.e., for $k=0,-1,-2,\ldots$, this gives

$$\chi_k(p)\sim|p-p_c|^{-\gamma} \tag{5.10a}$$

and for $k>\bar{\bar{d}}/2$, i.e., for $k=1,2,\ldots$,

$$\chi_k(p)\sim|p_c-p|^{-\gamma}\tau^{\bar{\bar{d}}/2-k}=|p-p_c|^{\beta-k\Delta_\tau} , \tag{5.10b}$$

as in Eq. (1.5).

Similarly, we can discuss $F_{\rm av}(t)$ defined in Eq. (1.3) within the fracton scaling picture. We write

$$F_{\rm av}(t)=\left[\sum_i\int_{\tau^{-1}}^1\rho_{ii}(\omega)e^{-\omega t}d\omega\right]_{\rm av} \tag{5.11a}$$

$$=|p-p_c|^{-\gamma}\int_{\tau^{-1}}^1\omega^{\bar{\bar{d}}/2-1}e^{-\omega t}d\omega . \tag{5.11b}$$

For t of order unity the exponential is of order unity

and if we combine the effect of all modes with frequencies of order unity, we may represent it as e^{-at} so that

$$F_{\rm av}(t) \sim |p-p_c|^{-\gamma} e^{-at}, \quad t \sim 1, \ a \sim 1 . \tag{5.12a}$$

For t of order τ, the dominant contribution from the integral over frequency comes from the region near the lower cutoff. Extending the high-frequency limit to infinity and changing variables of integration we write Eq. (5.11b) as

$$F_{\rm av}(t) \sim |p-p_c|^{-\gamma} \tau^{-\bar{d}/2} \int_1^\infty e^{-yt/\tau} y^{\bar{d}/2-1} dy , \tag{5.12b}$$

$$\sim |p-p_c|^{\beta} e^{-t/\tau}, \quad t \sim \tau . \tag{5.12c}$$

This form is schematic in that it gives the exponents in Eq. (1.5) correctly. However, by neglecting the fact that the frequencies have a distribution, it does not allow a discussion of the amplitudes A_k in Eq. (1.5). Thus, Eqs. (5.12a) and (5.12c) give a reasonable representation of $F_{\rm av}(t)$ both in the short-time domain, where the behavior depends in detail on the local modes, and in the long-time limit, where the scaling form of Eq. (1.2) holds.

VI. CONCLUSIONS

We may summarize our results as follows.

(1) We have obtained exact relations between moments of diffusion times and resistive correlations which are valid for arbitrary networks and can be extended to cover a wide class of models of diffusion, including the blind and myopic ants.

(2) Using these relations we have obtained exact solutions for the diffusional susceptibilities $\chi_k(p)$ (the average over clusters of moments of the diffusion times) for the Cayley tree and also in one dimension.

(3) Using scaling arguments we have shown that the critical exponents for $\chi_k(p)$ obey a gap relation: $\chi_k(p) \sim |p-p_c|^{\beta-k\Delta_\tau}$, where $\Delta_\tau=\beta+\gamma+\zeta$, where β and γ are percolation exponents and ζ is the resistivity scaling exponent. We then expect that suitable amplitude ratios will assume universal values independent of lattice structure, range of interaction (at least as long as the range is finite), etc.

(4) Using series expansions, we studied $\chi_k(p)$ and confirmed the conclusions in item (3). In particular, the values of the gap exponent agree very well with the accepted values of $\beta+\gamma+\zeta$ and the amplitude ratios we find agree in the limit of one dimension and in the limit of high dimension with our exact calculations mentioned in item (2).

(5) Our conclusions regarding corrections to scaling are less definite. In the exact results, we have been able to obtain, the analytic corrections for the myopic ant are somewhat larger than for the blind ant. It would be interesting to give a more complete analysis within the renormalization-group ϵ expansion. We have formulated the resistive correlations in field theoretic terms, but the subsequent analysis is not yet complete.

(6) The present work does suggest renormalization group calculations which ought now to be done. Firstly, it would be of interest to calculate for, say the blind ant, the amplitude ratios whose values are given in Table III. In principle the formalism for doing this exists.[25] Secondly, one should analyze the additional singular corrections for the myopic ant whose presence is indicated by Eqs. (2.34).

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APPENDIX A: EXACT RESULTS FOR THE CAYLEY TREE

In this appendix we give exact results for some diffusional susceptibilities on the Cayley tree. We con-

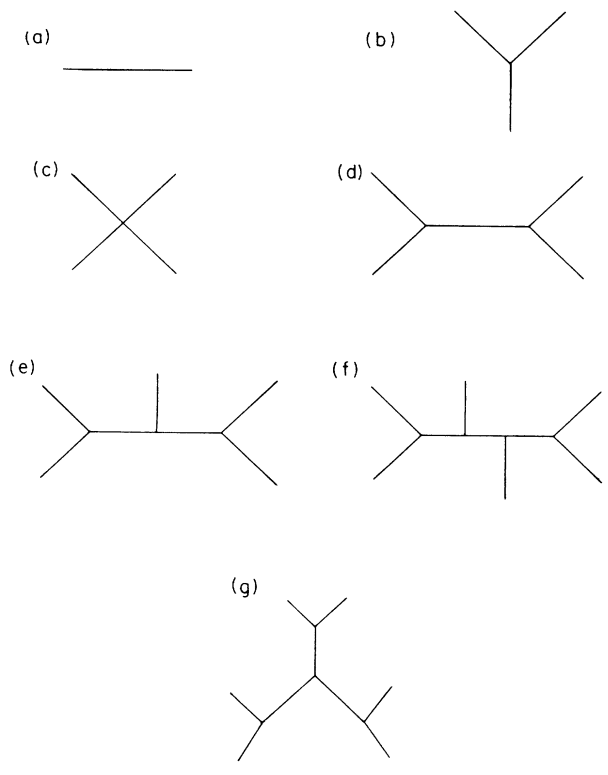

FIG. 1. Diagrams used for Cayley tree calculation. Here each line segment represents an arbitrarily long linear chain of bonds. Summing Eq. (A2) over diagrams having the topology shown yields contributions to correlation functions of the form $D(\sigma)[(\sigma+1)/\sigma]f(\sigma p)$, where $D(\sigma)$ is 1, $(\sigma-1)/\sigma$, $(\sigma-1)(\sigma-2)/\sigma^2$, $(\sigma-1)^2/\sigma^2$, $(\sigma-1)^3/\sigma^3$, $(\sigma-1)^4/\sigma^4$, and $(\sigma-1)^4/\sigma^4$, respectively, for topologies of the types a, b, c, d, e, f, and g, shown here.

sider first the blind ant. From Eq. (2.21) we get

$$\chi_1(p)=\sum_j [R_{ij}]_p = zp(1-\sigma p)^{-2} , \tag{A1}$$

where $\sigma = z-1$. Here $[\ ]_p$ denotes an average over all 2^N configurations of the lattice where N is the total number of bonds. This average should be distinguished from the average over clusters. If f_{ij} is a two-point function, then $\sum_j [f_{ij}]_p = [f(\Gamma)/s(\Gamma)]_{\rm av}$, where $f(\Gamma)=\sum_{i,j\in\Gamma} f_{ij}$. To evaluate averages over configurations we use cumulants, writing

$$[X]_p = \sum_\Gamma W(\Gamma) X_c(\Gamma) p^{s(\Gamma)} , \tag{A2}$$

where X is some quantity to be averaged and $X_c(\Gamma)$ is its cumulant value, defined recursively by

$$X_c(\Gamma) = X(\Gamma) - \sum_{\gamma\subset\Gamma} X_c(\gamma) , \tag{A3}$$

where the sum is over all γ which are subsets of Γ, with $\gamma=\Gamma$ excluded.

When X is a product of resistances, R_{ij} and/or connectivities v_{kl} [see Eq. (A4), below], then the cumulant value vanishes if a free end of Γ is not in the subset of indices of the R_{ij}'s or v_{ij}'s. Here v_{ij} is unity if sites i and j are in the same cluster and is zero otherwise. Thus, for example, in deriving Eq. (A1), only chain diagrams (with i and j at opposite ends of the chain) give nonzero contributions in Eq. (A2) as applied to $[R_{ij}]_p$.

To evaluate $\chi_2(p)$ for the blind ant we use

$$\left[\sum_j R_{ij}^2\right]_p = (\sigma+1)p(1+\sigma p)(1-\sigma p)^{-3} , \tag{A4a}$$

$$\left[\sum_{j,k} R_{ij}R_{ik}\right]_p = (\sigma^2-1)\sigma p^3(4+\sigma p)(1-\sigma p)^{-5} + (\sigma+1)p(1+5\sigma p+\sigma^2p^2)(1-\sigma p)^{-4} , \tag{A4b}$$

$$\left[\sum_{j,k,l} R_{ij}v_{jk}R_{kl}\right]_p = \frac{(\sigma+1)p(2+18\sigma p+18\sigma^2p^2+2\sigma^3p^3)}{(1-\sigma p)^5} + \frac{(\sigma^2-1)\sigma p^3(24+36\sigma p+4\sigma^2p^2)}{(1-\sigma p)^5} + \frac{4(\sigma+1)\sigma(\sigma-1)(\sigma-2)p^4}{(1-\sigma p)^6} + (\sigma^2-1)(\sigma-1)\sigma^2p^5\frac{22+2\sigma p}{(1-\sigma p)^7} . \tag{A4c}$$

The terms in Eqs. (A4) come from summing contributions in Eq. (A2) from diagrams of types a, b, c, and d in Fig. 1.

To evaluate $\chi_2(p)$ via Eq. (2.22) we invoke the relation[26] for the Cayley tree

$$[s(\Gamma)X(\Gamma)]_{\rm av} = (1-\sigma p)^{-1}[p(1-p)(\partial/\partial p)+(1+p)][X(\Gamma)]_{\rm av} . \tag{A5}$$

This relation can be integrated to give $[s^{-1}(\Gamma)X(\Gamma)]_{\rm av}$ in terms of $[X(\Gamma)]_{\rm av}$. Applying Eq. (A5) to the left-hand side of Eq. (A4a) and the inverse of (A5) to the left-hand side of Eq. (A4c) and combining these results with Eq. (A4b) we get the exact result for $\chi_2(p)$ for the Cayley tree:

$$\begin{aligned}\chi_2(p) = {} & \frac{2(1-p)(\sigma+1)^2}{(\sigma-1)^5} + \frac{(2-p)(\sigma+1)^2}{(\sigma-1)^4} - \frac{2\sigma(\sigma+1)^2(1-p)^2}{p(\sigma-1)^6}\ln\frac{(1-p)}{(1-\sigma p)} \\ & + \frac{(\sigma+1)(1-p)^2\sigma p}{(\sigma-1)^3(1-\sigma p)^2}(\sigma^2-3\sigma+4) + \frac{(\sigma+1)(1-p)^2\sigma^2p^2}{3(\sigma-1)^3(1-\sigma p)^3}(38-54\sigma+20\sigma^2) \\ & + \frac{(\sigma+1)(1-p)^2\sigma^3p^3}{2(\sigma-1)^2(1-\sigma p)^4}(21\sigma-27) + \frac{24(\sigma+1)(1-p)^2\sigma^4p^4}{5(\sigma-1)(1-\sigma p)^5} - \frac{2p(\sigma+1)}{(1-\sigma p)^4}(1+5\sigma p+\sigma^2p^2) \\ & - (8+2\sigma p)\frac{(\sigma^2-1)\sigma p^3}{(1-\sigma p)^5} + \frac{(\sigma+1)}{(1-\sigma p)^4}[p(1-p)(1+2\sigma p)+p(1+p)(1+\sigma p)] \\ & + \frac{(\sigma+1)}{(1-\sigma p)^5}[3\sigma p^2(1-p)(1+\sigma p)] .\end{aligned} \tag{A6a}$$

It is hard to imagine obtaining this result directly, i.e., without using Eq. (A5). For $\sigma=1$ (one dimension) this result becomes

$$\chi_2(p) = 2y + \tfrac{22}{3}y^2 + \tfrac{32}{3}y^3 + \tfrac{16}{3}y^4 , \tag{A6b}$$

where $y=p/(1-p)$. The result, Eq. (A6b) agrees with what one gets by substituting Eq. (3.2b) into Eq. (3.1) and keeping all the correction terms.

Clearly it will be tedious to get an exact expression for $\chi_3(p)$. However, to calculate the lowest amplitude ratio, $R_{13/22}=\chi_1(p)\chi_3(p)/[\chi_2(p)]^2$ evaluated at $p=p_c$, we only need the most divergent terms in $\chi_3(p)$ for $p\to p_c$. For a quantity with k indices, the dominant behavior (for $\sigma>1$) comes from diagrams with the maximal number of "links." For quantities with 3, 4, and 5 indices the dominant behavior is found by summing over diagrams of Fig. 1 of the respective types b, d, and e, and for 6 indices over types f and g. These "maximal" diagrams are similar to the mean-field diagrams in the calculation of

Stephen and Aronovitz[27] for polymer shapes.

To illustrate these arguments we calculate, from the diagram of type b, that

$$\sum_{i,j}[R_{ij}R_{jk}R_{ki}]_p \sim \sum_{l,m,n}(l+m)(m+n)(n+l)(\sigma p)^{l+m+n}\frac{(\sigma^2-1)}{\sigma^2} . \tag{A7}$$

To write this we summed over diagrams consisting of links of l, m, and n bonds and evaluated the cumulant as being the value of $R_{ij}R_{jk}R_{ki}$ with the sites i, j, and k on the free ends. Then, keeping only the dominant terms for $p \to p_c$ we have

$$[A]_{\rm av} \equiv \sum_{j,k}[R_{ij}R_{jk}R_{ki}]_p \sim 14(\sigma^2-1)\sigma^{-2}(1-\sigma p)^{-6} . \tag{A8}$$

Similarly, summing over diagrams of type d gives

$$[B]_{\rm av} \equiv \sum_{j,k,l}[R_{ij}R_{jk}R_{kl}]_p \sim 86(\sigma+1)(\sigma-1)^2\sigma^{-3}(1-\sigma p)^{-8} \tag{A9a}$$

over diagrams of type e

$$[C]_{\rm av} \equiv \sum_{j,k,l,m}[R_{ij}R_{jk}\nu_{kl}R_{lm}]_p \sim 664(\sigma+1)(\sigma-1)^3\sigma^{-4}(1-\sigma p)^{-8} , \tag{A9b}$$

and over diagrams of types f and g

$$[D]_{\rm av} \equiv \sum_{j,k,l,m,n}[R_{ij}\nu_{jk}R_{kl}\nu_{lm}R_{mn}]_p \sim 6376(\sigma+1)(\sigma-1)^4\sigma^{-5}(1-\sigma p)^{-12} . \tag{A9c}$$

Using Eq. (A5) or its inverse and keeping only the dominant terms we get

$$[s(\Gamma)A]_{\rm av} = 84K(1-\sigma p)^{-8} ,$$
$$[s(\Gamma)^{-1}C]_{\rm av} = 83K(1-\sigma p)^{-8} , \tag{A10}$$
$$[s(\Gamma)^{-2}D]_{\rm av} = \tfrac{797}{10}K(1-\sigma)^{-8} ,$$

where $K=(\sigma+1)(\sigma-1)^2/\sigma^3$. Inserting the above results into Eqs. (2.21)–(2.23) we get (for $\sigma>1$)

$$\chi_1(p) \sim (\sigma+1)\sigma^{-1}(1-\sigma p)^{-2} , \tag{A11a}$$

$$\chi_2(p) \sim \tfrac{4}{5}(\sigma+1)(\sigma-1)\sigma^{-2}(1-\sigma p)^{-5} , \tag{A11b}$$

$$\chi_3(p) \sim \tfrac{47}{10}(\sigma+1)(\sigma-1)^2\sigma^{-3}(1-\sigma p)^{-8} , \tag{A11c}$$

so that $R_{13/22}=\frac{235}{32}$. For $\sigma=1$ the asymptotic behavior is given by Eq. (3.3).

For the myopic ant we give only a few results for the Cayley tree. To avoid complications from factors of $\sum_i z_i = 2b(\Gamma)$ in the denominators we give results for quantities multiplied by suitable factors of $b(\Gamma)$. For instance, we have

$$\tilde{\chi}_1^M(p) \equiv \left[[b(\Gamma)/s(\Gamma)]\sum_n{}' \mu_n^{-1}\right]_{\rm av} = \tfrac{1}{4}\left[\sum_j R_{ij}z_iz_j\right]_p = \frac{(\sigma+1)p}{4}\left[\frac{1+\sigma p}{1-\sigma p}\right]^2 , \tag{A12a}$$

$$\begin{aligned}\tilde{\chi}_2^M(p) &\equiv \left[[b^2(\Gamma)/s(\Gamma)]\sum_n{}' \mu_n^{-2}\right]_{\rm av} \\ &= \tfrac{1}{4}\left[\sum_j R_{ij}z_iz_jb^2(\Gamma) - \sum_{j,k}R_{ij}R_{ik}z_iz_jz_kb(\Gamma) + \tfrac{1}{4}\sum_{j,k,l}R_{ij}R_{kl}z_iz_jz_kz_l\nu_{jk}\right]_p \\ &= \frac{(\sigma+1)p}{8}\left[\frac{128}{(1-\sigma p)^5} - \frac{320}{(1-\sigma p)^4} + \frac{288}{(1-\sigma p)^3} - \frac{112}{(1-\sigma p)^2} + \frac{18}{1-\sigma p} - 1\right] \\ &\quad + (\sigma^2-1)\sigma p^3\left[\frac{16}{(1-\sigma p)^6} - \frac{24}{(1-\sigma p)^5} + \frac{10}{(1-\sigma p)^4} - \frac{1}{(1-\sigma p)^3}\right] \\ &\quad + (\sigma^2-1)\sigma^2(\sigma-1)p^5[4(1-\sigma p)^{-7} - 2(1-\sigma p)^{-6}] .\end{aligned} \tag{A12b}$$

From these we get the asymptotic results

$$\left[[b(\Gamma)/s(\Gamma)]\sum_n{}' \mu_n^{-1}\right]_{\rm av} \sim (\sigma+1)\sigma^{-1}(1-\sigma p)^{-2} , \tag{A13a}$$

$$\left[[b(\Gamma)/s(\Gamma)]^2\sum_n{}' \mu_n^{-2}\right]_{\rm av} \sim \tfrac{4}{5}(\sigma+1)(\sigma-1)\sigma^{-2}(1-\sigma p)^{-5} . \tag{A13b}$$

Note that these are exactly comparable to Eqs. (A11a) and (A11b). Since we expect $s(\Gamma)$ and $b(\Gamma)$ to be identical from a scaling point of view, we expect the left-hand sides of Eq. (A13) to be asymptotically equivalent to $[b(\infty)/s(\infty)]^k\chi_k^M(p)$ as defined in Eq. (2.10). This result is therefore consistent with our statement that the blind and myopic ants are asymptotically equivalent. For the Cayley tree, z_i can be replaced in the dominant term by a constant which we find to be

$2b(\infty)/s(\infty)=2$. This constant will be different in finite spatial dimensions, and in fact we argued below Eq. (2.34a) that its value is nonuniversal. We should note that the Cayley tree results given in Eq. (A13) are valid for $\sigma>1$. For the linear chain ($\sigma=1$) the results are as given in Sec. III.

APPENDIX B: χ_k FOR THE MYOPIC ANT IN TERMS OF RESISTANCES

Here we derive the relation between the higher-order diffusional susceptibilities for the myopic ant and resistance correlations. Our approach is to calculate $G^M(\omega)$ from Eq. (2.29) by perturbation theory in ωZ. Let $\hat{\lambda}_n(\omega)$ denote the exact eigenvalues of $M^B+\omega Z$ and $|\hat{n}\rangle$ their corresponding eigenvectors. Then we may use Eqs. (2.29) and (2.30) to write

$$\hat{G}^M(\omega)\equiv G^M(\omega)-\omega^{-1}=\sum_n\left[\langle\hat{n}\,|\,Z\,|\,\hat{n}\rangle-\frac{\langle 0\,|\,Z\,|\,\hat{n}\rangle^2}{\langle 0\,|\,Z\,|\,0\rangle}\right]\hat{\lambda}_n^{-1}(\omega)\ . \tag{B1}$$

Treating ωZ as a perturbation, we have

$$\hat{\lambda}_n(\omega)=\lambda_n+\omega\langle n\,|\,Z\,|\,n\rangle+\omega^2\sum_{\substack{m\\(m\neq n)}}\frac{\langle m\,|\,Z\,|\,n\rangle^2}{\lambda_n-\lambda_m}\ , \tag{B2a}$$

$$|\hat{n}\rangle=|n\rangle+\omega\sum_{\substack{m\\(m\neq n)}}(|m\rangle\langle m\,|\,Z\,|\,n\rangle)/(\lambda_n-\lambda_m)\ , \tag{B2b}$$

where λ_n and $|n\rangle$ are the eigenvalues and eigenvectors of M^B. Correct to order ω, Eq. (B1) is

$$\begin{aligned}\hat{G}^M(\omega)=\omega\sum_m{}'\sum_r{}'\frac{\langle 0\,|\,Z\,|\,m\rangle\langle m\,|\,Z\,|\,r\rangle\langle r\,|\,Z\,|\,0\rangle}{\langle 0\,|\,Z\,|\,0\rangle\lambda_m\lambda_r}-\omega\left[\sum_m{}'\frac{\langle 0\,|\,Z\,|\,m\rangle^2}{\langle 0\,|\,Z\,|\,0\rangle\lambda_m}\right]^2\\ +\sum_n{}'\left[\langle n\,|\,Z\,|\,n\rangle-\frac{\langle 0\,|\,Z\,|\,n\rangle^2}{\langle 0\,|\,Z\,|\,0\rangle}+2\omega\sum_{\substack{m\\(m\neq n)}}\frac{\langle n\,|\,Z\,|\,m\rangle^2}{\lambda_n-\lambda_m}-2\omega\sum_{\substack{m\\(m\neq n)}}\frac{\langle 0\,|\,Z\,|\,m\rangle\langle m\,|\,Z\,|\,n\rangle\langle n\,|\,Z\,|\,0\rangle}{(\lambda_n-\lambda_m)\langle 0\,|\,Z\,|\,0\rangle}\right]\\ \times(\lambda_n+\omega\langle n\,|\,Z\,|\,n\rangle)^{-1}\ ,\end{aligned} \tag{B3}$$

where the first two terms are contributions from $n=0$ in Eq. (B1). We wish to express the results in terms of the blind-ant Green's function $\hat{G}$, which in this notation is

$$\hat{G}=\sum_n{}'\,|n\rangle\lambda_n^{-1}\langle n\,|\ , \tag{B4}$$

which can be expressed in terms of resistances via Eq. (2.20). Expanding Eq. (B3) in powers of ω, one gets Eq. (2.31b) from the constant term. The term linear in ω gives the result

$$\begin{aligned}\left.\frac{\partial\hat{G}}{\partial\omega}\right|_{\omega=0}=\frac{\langle 0\,|\,Z\hat{G}Z\hat{G}Z\,|\,0\rangle}{\langle 0\,|\,Z\,|\,0\rangle}-\left[\frac{\langle 0\,|\,Z\hat{G}Z\,|\,0\rangle}{\langle 0\,|\,Z\,|\,0\rangle}\right]^2+2\sum_n{}'\frac{\langle n\,|\,Z\,|\,0\rangle^2}{\lambda_n^2}\\ +2\sum_n{}'\sum_m{}'(1-\delta_{n,m})\frac{\langle n\,|\,Z\,|\,m\rangle^2}{\lambda_n(\lambda_n-\lambda_m)}-2\sum_n{}'\frac{\langle n\,|\,Z\,|\,0\rangle^2}{\lambda_n^2}\\ -2\sum_n{}'\sum_m{}'(1-\delta_{n,m})\frac{\langle 0\,|\,Z\,|\,m\rangle\langle m\,|\,Z\,|\,n\rangle\langle n\,|\,Z\,|\,0\rangle}{\lambda_n(\lambda_n-\lambda_m)\langle 0\,|\,Z\,|\,0\rangle}-\sum_n{}'\left[\langle n\,|\,Z\,|\,n\rangle-\frac{\langle 0\,|\,Z\,|\,n\rangle^2}{\langle 0\,|\,Z\,|\,0\rangle}\right]\frac{\langle n\,|\,Z\,|\,n\rangle}{\lambda_n^2}\ .\end{aligned} \tag{B5}$$

In the double sums we have written the $m=0$ terms separately so that in these sums the summands can be replaced by their symmetrized (in n and m) values. Thus we make the replacement

$$\frac{2}{\lambda_n(\lambda_n-\lambda_m)}\to\frac{1}{\lambda_n-\lambda_m}\left[\frac{1}{\lambda_n}-\frac{1}{\lambda_m}\right]=-\frac{1}{(\lambda_n\lambda_m)}\ . \tag{B6}$$

After this manipulation the terms involving $\delta_{n,m}$ in the double sums in Eq. (B5) cancel the last term, with the result that

$$-\left.\frac{\partial\hat{G}}{\partial\omega}\right|_{\omega=0}=\mathrm{Tr}\hat{G}Z\hat{G}Z-2\frac{\langle 0\,|\,Z\hat{G}Z\hat{G}Z\,|\,0\rangle}{\langle 0\,|\,Z\,|\,0\rangle}+\left[\frac{\langle 0\,|\,Z\hat{G}Z\,|\,0\rangle}{\langle 0\,|\,Z\,|\,0\rangle}\right]^2\ . \tag{B7}$$

Using Eq. (2.20) we express the result in terms of resis-

tances as

$$-\frac{\partial \hat{G}}{\partial \omega}\bigg|_{\omega=0} = \tfrac{1}{4}\sum_{i,j} R_{ij}^2 z_i z_j + \tfrac{1}{4}\left[\sum_{i,j} z_i R_{ij} z_j \Big/ \sum_i z_i\right]^2 - \tfrac{1}{2}\sum_{i,j,k} R_{ij}R_{jk} z_i z_j z_k \Big/ \sum_i z_i \ . \tag{B8}$$

When averaged over clusters this result leads to Eq. (2.34b). To calculate higher-order moments requires carrying the expansion in Eqs. (B1) and (B2) to higher order in ω. Such a calculation to order ω^2 gives

$$\frac{1}{2}\frac{\partial^2 \hat{G}}{\partial \omega^2}\bigg|_{\omega=0} = \mathrm{Tr}\hat{G}Z\hat{G}Z\hat{G}Z - 3\frac{\langle 0 \mid Z\hat{G}Z\hat{G}Z\hat{G}Z \mid 0\rangle}{\langle 0 \mid Z \mid 0\rangle} - \left[\frac{\langle 0 \mid Z\hat{G}Z \mid 0\rangle}{\langle 0 \mid Z \mid 0\rangle}\right]^3 + 3\frac{\langle 0 \mid Z\hat{G}Z\hat{G}Z \mid 0\rangle\langle 0 \mid Z\hat{G}Z \mid 0\rangle}{\langle 0 \mid Z \mid 0\rangle^2} \ . \tag{B9}$$

In terms of resistances this is

$$\frac{1}{2}\frac{\partial^2 \hat{G}}{\partial \omega^2}\bigg|_{\omega=0} = -\tfrac{1}{8}\sum_{i,j,k} R_{ij}R_{jk}R_{ki} z_i z_j z_k + \tfrac{3}{8}\sum_{i,j,k,l} R_{ij}R_{jk}R_{kl} z_i z_j z_k z_l \Big/ \sum_i z_i - \tfrac{3}{8}\left[\sum_{i,j} R_{ij} z_i z_j\right]\left[\sum_{i,j,k} R_{ij}R_{jk} z_i z_j z_k\right] \Big/ \left[\sum_i z_i\right]^2 + \tfrac{1}{8}\left[\sum_{i,j} R_{ij} z_i z_j \Big/ \sum_i z_i\right]^3 \ . \tag{B10}$$

When averaged over clusters, this leads to Eq. (2.34c).

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